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POTENTIALS AND ISOMETRIC EMBEDDINGS IN L_1

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ABSTRACT

l_p^3 does not embed isometrically in L_1, for $p > 2$. The question when l_r^n embeds isometrically in L_p is completely answered.

It is well known that L_p embeds in L_1 isometrically, if $1 \leqq p \leqq 2$ (see, e.g., [2]). E. D. Bolker [1] conjectured that l_p^3 does not embed isometrically in L_1 for $p > 2$. We substantiate this conjecture here. (In the case $p > 2.7$, this has been done by H. S. Witsenhausen [11].) The result we prove (Theorem 1.5) actually implies that if E is a Banach space not isometric to a Hilbert space, and if $p > 2$, then the direct sum of E with the real line in the l_p sense does not embed isometrically in L_1.

This should be contrasted with the recent result of R. Schneider [10], who showed that for any $n \geqq 2$ there is an n-dimensional Banach space E such that both E and E^* embed isometrically in L_1, but E is not isometric to l_2^n. Schneider's result solved a question of Grothendieck [4], and is striking in view of the analogous isomorphic question: Grothendieck [4, p. 66] proved that if E and E^* embed isomorphically in L_1 then E is isomorphic to a Hilbert space.

In Section 1 we first give an inversion formula for the potential transform on the line, and characterize the functions which are potentials of bounded positive measures on **R**. Then we show how potentials arise naturally in the study of isometric embeddings of two-dimensional spaces in L_1, and prove that such embeddings are unique in a certain sense. Finally we use these results to prove our main result.

In Section 2 we show that for $1 < p < \infty$, l_r^2 embeds isometrically in L_p if and only if $1 < p \leqq r \leqq 2$ or $r = 2$.

L_p denotes the space of pth power integrable functions on $[0,1]$ with $\| f \| = (\int_0^1 | f(x) |^p dx)^{1/p}$, l_p^n denotes $\mathbf{R}^n$ with the norm $\|(a_i ; i \leqq n)\| = (\Sigma_{i=1}^n | a_i |^p)^{1/p}$.

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For a set A, 1_A is its indicator function, and A is its complement. All the Banach spaces considered are over the reals.

1. Embeddings in L_1

PROPOSITION 1.1. *Let μ be a finite positive measure on* **R** *satisfying*

$$\int_{-\infty}^{\infty} |x|\, d\mu(x) < \infty. \tag{1}$$

Define a function φ on **R** *by*

$$\varphi(t) = \int_{-\infty}^{\infty} |x - t|\, d\mu(x), \qquad t \in \mathbf{R}. \tag{2}$$

Then φ is a convex positive function on **R** *s.t.*

$$\lim_{|t|\to\infty} \frac{\varphi(t)}{|t|} = \|\mu\| \equiv \mu(\mathbf{R}) < \infty, \quad \text{and} \tag{3}$$

$$\lim_{t\to\infty}\{t\varphi'_+(t) - \varphi(t)\} = -\lim_{t\to-\infty}\{t\varphi'_+(t) - \varphi(t)\} \quad \text{is finite.} \tag{4}$$

This function φ determines uniquely μ by the relations (3) *and*

$$\mu((-\infty, t]) = \tfrac{1}{2}(\|\mu\| + \varphi'_+(t)). \tag{5}$$

Conversely, if φ is a convex function satisfying (3) *and* (4), *then* (3) *and* (5) *define a finite positive measure μ on* **R** *satisfying* (1) *and* (2).

PROOF. $\varphi(t)$ is a positive convex function of t, since $|x - t|$ is. Dominated convergence theorem yields (3) and the possibility to differentiate (2) under the integral sign, which gives

$$\varphi'_+(t) = \int_{-\infty}^{\infty} H_t(x)\, d\mu(x) = \mu((-\infty, t]) - \mu((t, \infty)), \tag{6}$$

where $H_t(x) = (1_{(-\infty,t]} - 1_{(t,\infty)})(x)$, which renders (5). Also, using (6) and the fact that $|x - t| = (t - x)H_t(x)$ we have

$$t\varphi'_+(t) - \varphi(t) = \int_{-\infty}^{\infty} xH_t(x)d\mu(x) \to \pm \int_{-\infty}^{\infty} x d\mu(x) \quad \text{as} \quad t \to \pm\infty,$$

which shows (4). Now let φ be convex on **R** and satisfy the conditions (3) and (4). Then $\varphi'_+(t)$ is well defined, increasing, and right-continuous on **R**. By (3), $\lim_{t\to\pm\infty} \varphi'_+(t) = \pm\|\mu\|$, so (5) defines a positive Borel measure on **R** with

$\mu(\mathbf{R}) = \|\mu\| < \infty$. Let us denote the limit on the left hand side of (4) by c. Integrating by parts, and noting that a convex function is absolutely continuous,

$$\int_t^b (x-t)\,d\varphi'_+(x) = [(x-t)\varphi'_+(x) - \varphi(x)]_{x=t}^b$$
$$= b\varphi'_+(b) - \varphi(b) - t\varphi'_+(b) + \varphi(t) \to c - t\|\mu\| + \varphi(t), \quad \text{as} \quad b \to +\infty.$$

Similarly,

$$\int_{-\infty}^t (t-x)\,d\varphi'_+(x) = -c + t\|\mu\| + \varphi(t) \quad \text{so} \quad \int_{-\infty}^{\infty} |t-x|\,d\varphi'_+(x) = 2\varphi(t),$$

proving (2) and (1). Q.E.D.

REMARKS. (i) The function $(-\varphi(t))$ is the classical one-dimensional potential of the measure μ, in (2). It has been shown [3] that the map $\mu \to \varphi$ is one-to-one, but we did not find in the literature the inversion formula (5) and the characterization of potentials given by (3) and (4).

(ii) Let $1 < p < \infty$, and consider $\varphi(t) = \int_{-\infty}^{\infty} |x-t|^p d\mu$, for suitable positive measures μ.

QUESTION. Is the map $\mu \to \varphi$ one-to-one?

The answer is "yes" for p an odd integer, and "no" for p an even integer. It seems plausible that the answer is "yes" for p not an integer.

LEMMA 1.2. *Let e_1, e_2 be a basis for a Banach space E over the reals. Define the function φ on $\mathbf{R}$ by*

$$\varphi(t) = \|e_1 - te_2\|, \qquad t \in R. \tag{7}$$

Then φ is a positive convex function satisfying

$$\lim_{|t|\to\infty} \frac{\varphi(t)}{|t|} \equiv c \in \mathbf{R}, \qquad \text{and} \tag{8}$$

$$-\lim_{t\to-\infty}\{\varphi(t) - t\varphi'_+(t)\} \leqq \lim_{t\to\infty}\{\varphi(t) - t\varphi'_+(t)\}. \tag{9}$$

Conversely, if φ is a positive convex function on $\mathbf{R}$ satisfying (8) and (9), then there is a unique norm $\|\cdot\|$ on E satisfying (7).

PROOF. Let φ be given by (7), and put

$$\psi(u) = \|e_2 - ue_1\|, \qquad u \in \mathbf{R}. \tag{10}$$

Then (8) holds with $c = \psi(0) = \|e_2\|$. For $u \neq 0$, $\psi(u) = |u|\,\varphi\left(\frac{1}{u}\right)$, so

$$\psi'_-(u) = (\operatorname{sgn} u)\varphi\left(\frac{1}{u}\right) - \frac{|u|}{u^2}\varphi'_+\left(\frac{1}{u}\right) = (\operatorname{sgn} u)\left\{\varphi\left(\frac{1}{u}\right) - \frac{1}{u}\varphi'_+\left(\frac{1}{u}\right)\right\},$$

and so

$$\psi'_-(0) = -\lim_{u\to 0-}\left\{\varphi\left(\frac{1}{u}\right) - \frac{1}{u}\varphi'_+\left(\frac{1}{u}\right)\right\} \leqq \psi'_+(0) = \lim_{u\to 0+}\left\{\varphi\left(\frac{1}{u}\right) - \left(\frac{1}{u}\right)\varphi'_+\left(\frac{1}{u}\right)\right\},$$

implying (9). Conversely, if $x = a_1e_1 + a_2e_2$ denotes a generic element of E, then (7) and the condition $\| \alpha x \| = |\alpha| \cdot \| x \|$ define the value of $\| x \|$ for $a_1 \neq 0$, and $\| \cdot \|$ is convex on each of the half-planes $\{x; a_1 > 0\}$ and $\{x; a_1 < 0\}$. Now, (10) defines a function $\psi(u)$ for $u \neq 0$. Putting $\psi(0) = c$ we obtain a continuous function ψ on $\mathbf{R}$. ψ is convex on each of $[0,\infty)$ and $(-\infty,0]$, and by (9), $\psi'_-(0) \leqq \psi'_+(0)$, so ψ is convex on all of $\mathbf{R}$. Therefore, putting $\| a_2e_2 \| = |a_2| \psi(0)$, we find that $\| \cdot \|$ is convex on each of the half-planes $\{x; a_2 > 0\}$ and $\{x; a_2 < 0\}$, and so $\| \cdot \|$ is convex on E. Q.E.D.

REMARK. If we put $\psi(u) = \int_{-\infty}^{\infty} |1 - ux| \, d\mu(x)$ in Proposition 1.1 analogously to (10), then (4) says that ψ is differentiable at 0, which corresponds to the fact that the L_1-norm is smooth at the point 1 (i.e. there is only one supporting hyperplane to the unit ball of L_1 at the point 1).

PROPOSITION 1.3. *Let e_1, e_2 be a basis for a Banach space E over the reals. There is a unique finite positive measure μ on $\mathbf{R}$ and a unique $b \geqq 0$ so that*

$$\| a_1e_1 + a_2e_2 \| = \int_{-\infty}^{\infty} |a_1x + a_2| \, d\mu(x) + b\,|a_1|, \qquad a_1, a_2 \in \mathbf{R}. \tag{11}$$

$b = 0$ iff $\| \cdot \|$ is smooth at e_2.

PROOF. Let $\varphi(t) = \| e_1 - te_2 \|$. If $\| \cdot \|$ is smooth at e_2, then we have equality in (9) and so Proposition 1.1 gives the representation (11) with $b = 0$. The integral in (11) is differentiable in a_1 at $a_1 = 0$, $a_2 = 1$, which proves the uniqueness of b and μ in this case. If $\| \cdot \|$ is not smooth at e_2 we have

$$2b \equiv \lim_{t\to\infty}\{\varphi(t) - t\varphi'_+(t)\} + \lim_{u\to-\infty}\{\varphi(u) - u\varphi'_+(u)\} > 0.$$

Let $\tilde{\varphi}(t) = \varphi(t) - b$. $\tilde{\varphi}$ is convex and satisfies (8) and (9), with equality in (9). The fact that $\tilde{\varphi} \geqq 0$ follows from the convexity of φ and from (8): if $u < v < t$, then

$$\frac{\varphi(t) - \varphi(v)}{t - v} \leqq \varphi'_+(t), \quad \frac{\varphi(v) - \varphi(u)}{v - u} \geqq \varphi'_+(u),$$

so

$$2\varphi(v) \geqq \varphi(t) - t\varphi'_+(t) + \varphi(u) - u\varphi'_+(u) + v(\varphi'_+(t) + \varphi'_+(u)).$$

Thus $2\varphi(v) \geqq 2b + v(\lim_{t\to\infty}\varphi'_+(t) + \lim_{u\to-\infty}\varphi'_+(u)) = 2b$, using (8). Applying the case already proved, we have a unique finite positive measure μ on $\mathbf{R}$ such that $\tilde{\varphi}(t) = \int_{-\infty}^{\infty}|x - t|\,d\mu(x)$. But $\varphi(t) = \tilde{\varphi}(t) + b$. Q.E.D.

REMARK. The proposition may be restated as follows: Let e_1, e_2 be a basis for a Banach space E. Then there is a measure λ and isometric embedding T of E into $L_1(\lambda)$. T may be taken such that $Te_2 = 1_A$ for some set A. In that case, $\lambda(A)$, $\int_{\sim A}|Te_1|\,d\lambda$ and the distribution of $(Te_1)_{|A}$ are uniquely determined by the norm of E.

COROLLARY 1.4 (See C. Herz [5], and J. Lindenstrauss [6]). *Every two-dimensional Banach space over the reals embeds isometrically in* $L_1(0,1)$.

We can now prove our main result:

THEOREM 1.5. *Let e_1, e_2 be a basis for a Banach space E such that $\|ue_1 + e_2\| = \|-ue_1 + e_2\|$, all $u \in \mathbf{R}$. If F is an arbitrary Banach space, $e \notin F$ an abstract element, let $F \oplus_E [e]$ denote the set of all expresions of the form $x + te$ with $x \in F$ and $t \in \mathbf{R}$, with the norm $\|x + te\| = \|(\|x\|)e_1 + te_2\|_E$. Assume that the function $\psi(u) = \|e_2 - ue_1\|$ is differentiable at 0 and satisfies the following "smoothness-type" condition for some $\varepsilon > 0$:*

$$q(u) = \psi(0) - \psi(u) + u\psi'(u) = O(u^{2+\varepsilon}), \quad \text{as} \quad u \to 0. \tag{12}$$

If $F \oplus_E [e]$ embeds isometrically in L_1, then F is isometrically isomorphic to the Hilbert space of its dimension.

PROOF. A significant part of this proof was suggested by R. P. Kaufman. We may assume that $\dim F = 2$, and that $\|e_2\|_E = 1$. Let $S_F = \{y \in F; \|y\| = 1\}$, and let T be an isometric linear embedding of $F \oplus_E [e]$ in $L_1(\lambda)$, λ a probability measure, T normalized so that $Te \equiv 1$. By Propositions 1.3 and 1.1, for any $y \in S_F$ we have

$$\lambda(\{\omega; (Ty)(\omega) \leqq t\}) = \tfrac{1}{2}\{1 + \varphi'_+(t)\}, \qquad t \in \mathbf{R}, \tag{13}$$

where $\varphi(t) = \|e_1 - te_2\|_E = \|y - te\|$. Thus all the functions Ty, $y \in S_F$ have the same distribution. Let $y, z \in S_F$ be a basis for F, and consider the joint characteristic function of Ty and Tz:

$$f(\xi, \eta) = \int \exp(i\xi(Ty) + i\eta(Tz))\,d\lambda, \qquad \xi, \eta \in \mathbf{R}.$$

For any $\xi, \eta, \| \xi y + \eta z \|_F^{-1}(\xi y + \eta z) \in S_F$, which implies

$$f(\xi, \eta) = f(\| \xi y + \eta z \|_F, 0), \ (\xi, \eta) \neq 0 \quad \text{in} \quad \mathbf{R}^2. \tag{14}$$

Accept for a moment the fact that f is twice continuously differentiable. We have $f_{\xi\xi}(0,0) = f_{\eta\eta}(0,0) \equiv A$, and let $f_{\xi\eta}(0,0) = B$. Then by the chain-rule, $\frac{\partial^2}{\partial \xi^2} f(\alpha\xi, \beta\xi)_{|\xi=0} = A\alpha^2 + 2B\alpha\beta + A\beta^2$, for any $\alpha, \beta \in \mathbf{R}$. On the other hand, $f(\alpha\xi, \beta\xi) = f(\xi \| \alpha y + \beta z \|_F, 0)$, by (14), so $\frac{\partial^2}{\partial \xi^2} f(\alpha\xi, \beta\xi)_{|\xi=0} = \| \alpha y + \beta z \|_F^2 \cdot A$. Thus $\| \alpha y + \beta z \|_F = \alpha^2 + 2(B/A)\alpha\beta + \alpha^2$, and so S_F is the linear image of an ellipse, i.e. F is isometrically isomorphic to the Euclidean plane.

To see that f is twice continuously differentiable it is enough to show that $\int (Ty)^2 d\lambda < \infty$, and apply the dominated convergence theorem. (Then $-A = \int (Ty)^2 d\lambda \neq 0$, which was used above.) By (13), we have $\lambda(\{\omega; |(Ty)(\omega)| \geqq t\}) = 1 - \frac{1}{2}\varphi'(t) + \frac{1}{2}\varphi'(-t) = 1 - \varphi'(t)$, wherever φ is differentiable (we use the fact that $\varphi(t) = \varphi(-t)$ by the assumptions of the theorem). Noting that $\varphi(t) = t\psi\left(\frac{1}{t}\right)$, it $t > 0$, and that $\psi(0) = 1$, we have $1 - \varphi'(t) = \psi(0) - \psi\left(\frac{1}{t}\right) + \frac{1}{t}\psi'\left(\frac{1}{t}\right) = q\left(\frac{1}{t}\right) = O(t^{-2-\varepsilon})$ by (12), and so $\Sigma_{n=1}^{\infty} \lambda\{\omega; |Ty|(\omega) \geqq \sqrt{n}\} < \infty$, which implies that $\int |Ty|^2 dy < \infty$ (see [8, p. 242]). Q.E.D.

COROLLARY. *l_p^3 does not embed isometrically in L_1, if $p > 2$.*

PROOF. Putting $E = F = l_p^2$, we have $\psi(u) = (1 + |u|^p)^{1/p}$, so $q(u) = 1 - (1 + |u|^p)^{(1/p)-1} = O(|u|^p)$, as $u \to 0$. Q.E.D.

REMARK. In connection with the proof of Theorem 1.5, R. P. Kaufman raised the following

QUESTION. Let $p > 2$. Does there exist a characteristic function $f(s, t)$ of two random variables such that $f(s,t) = f((|s|^p + |t|^p)^{1/p}, 0)$ for all $s, t \in \mathbf{R}$.

It is shown in [2] that there is no infinite sequence $X_1, X_2, \cdots$ of random variables such that the characteristic function $f(t_1, \cdots, t_n)$ of $X_1, \cdots, X_n$ depends only on $|t_1|^p + |t_2|^p + \cdots + |t_n|^p$, and $p > 2$.

2. Embeddings in L_p

We can now answer the question when does l_r^n embed isometrically in L_p.

THEOREM 2.1. *Let $1 \leqq p < \infty$, $1 \leqq r \leqq \infty$. The only cases when l_r^n embeds isometrically in $L_p = L_p(0,1)$ are those included in one of the following facts:*

a) *L_r embeds isometrically in L_p for $1 \leqq p \leqq r \leqq 2$.*

b) *L_2 and L_p embed isometrically in L_p, for all p.*

c) *Every two-dimensional Banach space embeds isometrically in L_1.*

PROOF. For the positive results see [2] and [5], [6], or Corollary 1.4 above. For $p = 1$ the negative result claimed is that l_r^3, $r > 2$ does not embed isometrically in L_1, which was proved in Theorem 1.5. Fix $1 < p < 2$, and assume that l_r^2 embeds isometrically in L_p for some $r \notin [p, 2]$. Let f, g be functions in L_p satisfying

$$\| \alpha f + \beta g \|_p = (|\alpha|^r + |\beta|^r)^{1/r}, \quad \text{all} \quad \alpha, \beta \in \mathbf{R}. \tag{15}$$

If $r < p$, then Clarkson's inequalities (see, e.g. [9, sec. 14.5]) give

$$2^{p/r} = \tfrac{1}{2}(\| f + g \|^p + \| f - g \|^p) \leqq \| f \|^p + \| g \|^p = 2 \tag{16}$$

which is a contradiction. If $2 < r < \infty$, consider

$$\varphi(t) = (1 + |t|^r)^{p/r} = \int_0^1 |f(\omega) + tg(\omega)|^p d\omega. \tag{17}$$

Using the first expression for φ we have $\varphi''(0) = 0$, since $r > 2$. In the second expression we may assume that $f \geqq 0$ almost everywhere. Let $A = \{\omega ; f(\omega) > 0\}$, and let $c = \int_{\sim A} |g(\omega)|^p d\omega$. $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1(t) = \int_A |f + tg|^p$, $\varphi_2(t) = c\,|t|^p$, $t \in \mathbf{R}$. We have $\varphi_1'(t) = p\int_A |f + tg|^{p-1} g$. By Fatou's Lemma we have

$$\varliminf_{t \to 0} \frac{\varphi_1'(t) - \varphi_1'(0)}{t} \geqq p \int_A \varliminf_{t \to 0} \frac{|f + tg|^{p-1} - f^{p-1}}{t} g = p(p-1) \int_A f^{p-2} g^2 \geqq 0.$$

If $c > 0$, we thus get $\varphi''(0) = +\infty$. If $c = 0$ we have $g_{|A} \neq 0$, and so

$$\varliminf_{t \to 0} \frac{\varphi'(t) - \varphi'(0)}{t} \geqq p(p-1) \int_A f^{p-2} g^2 > 0.$$

Both cases contradict the fact that $\varphi''(0) = 0$. Finally, l_∞^2 is not smooth.

Now fix $2 < p < \infty$, and assume that l_r^2 embeds in L_p for some $r \neq p, 2$. Let f, g be functions satisfying (15). If $r > p$, we obtain a contradiction by reverting the inequality in (16). For $2 \neq r < p$, we keep the definition (17) and the notation introduced subsequently. If $1 < r < 2$, then the first expression in (17) gives easily $\varphi''(0) = +\infty$, while the second one gives $\varphi''(0) = \int_A f^{p-2} g^2 + c \cdot 0 < \infty$. If $2 < r < \infty$, we have $\varphi''(0) = 0$ from the first equation in (17) and $\varphi''(0) = \int_A f^{p-2} g^2$ from the second one, forcing $g_{|A} = 0$. But then (17) gives $(1 + |t|^r)^{1/r} = (1 + |t|^p)^{1/p}$, which is absurd.

Finally note that l_1^2 and l_∞^2 do not embed in L_p, $1 < p < \infty$ since the latter is smooth. Q.E.D.

REMARKS. (i) These results show complete analogy to the isomorphic theory (cf., e.g. [7, II.3.c.]). In fact, we can say that l_r embeds isomorphically in L_p iff L_r embeds isometrically in L_p iff l_r^3 embeds isometrically in L_p.

ii) We have considered here only *linear* isometric embeddings, which is however no loss of generality, since every isometric embedding in a strictly convex normed space is affine, and since by [2] a normed space embeds isometrically in L_1 iff it embeds there linearly and isometrically. (E is strictly convex if $\| x + y \| = \| x \| + \| y \|$ implies $x = 0$ or $y = \alpha x$, some $\alpha \geqq 0$; T is affine if $x \to Tx - T0$ is linear.)

(iii) The differentiation under the integral sign was justified in each case by the dominated convergence theorem and Holder's inequality.

Added in proof. While this paper was in press, Professor H. Porta brought to my attention the recent paper of W. Rudin, *L^P-isometries and equimeasurability*, Indiana Univ. Math. J. **25** (1976), 215–228, in which the uniqueness part of Proposition 1.1 is proved in more generality, answering also (in the positive) the second remark to that Proposition. Rudin's proof, while covering the complex case, is not constructive. Note that the complex analogue of Corollary 1.4 is false (e.g., complex l_∞^2 does not embed isometrically in complex L_1).

More applications to the structure of subspaces of L_1 will be published elsewhere.

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